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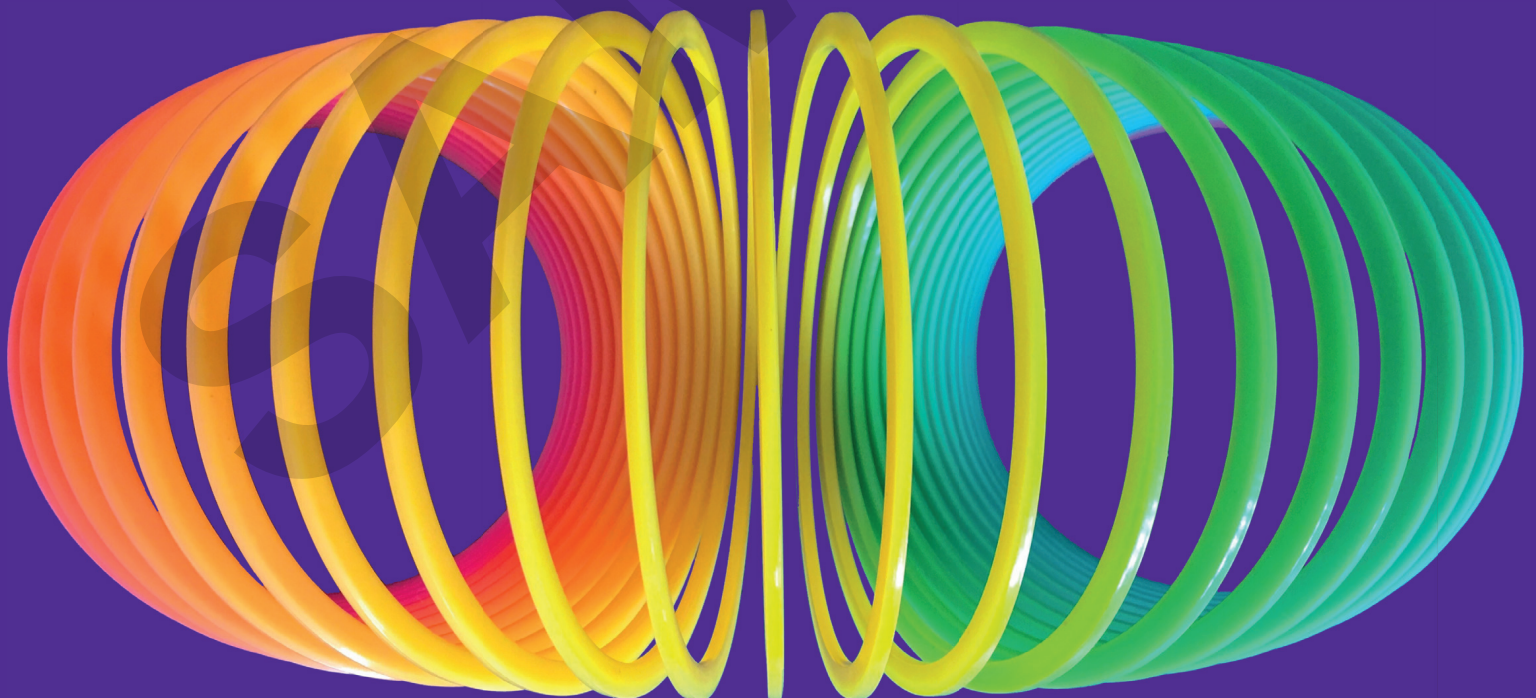


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Cambridge IGCSE™ and O Level Additional Mathematics

COURSEBOOK

Sue Pemberton



Third edition

Digital access



Cambridge Assessment
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Endorsed for full syllabus coverage

SAMPLE

Low-res

Chapter 1

Functions

Cambridge
University Press

THIS SECTION WILL SHOW YOU HOW TO:

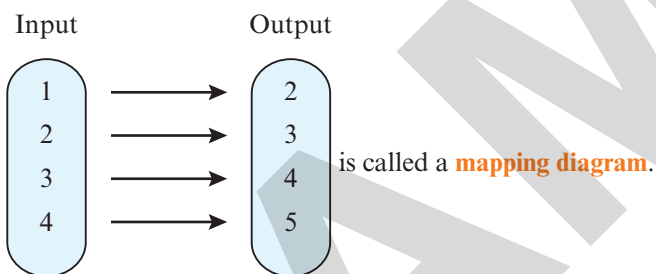
- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f: x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- solve graphically or algebraically equations of the type $|ax + b| = c$ and $|ax + b| = cx + d$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- sketch graphs to show the relationship between a function and its inverse.

PRE-REQUISITE KNOWLEDGE

Before you start...

Where it comes from	What you should be able to do	Check your skills
Cambridge IGCSE/O Level Mathematics	Find an output for a given function.	1 If $f(x) = 5x - 1$, find $f(3)$.
Cambridge IGCSE/O Level Mathematics	Find a composite function.	2 If $f(x) = 3x - 2$ and $g(x) = 4 - x$, find $fg(x)$.
Cambridge IGCSE/O Level Mathematics	Find the inverse of a simple function.	3 If $f(x) = 3x + 5$, find $f^{-1}(x)$.
Cambridge IGCSE/O Level Mathematics	Sketch linear and quadratic graphs.	4 a Sketch the graph of $y = 2x - 1$. b Sketch the graph of $y = x^2 + 1$.
Cambridge IGCSE/O Level Mathematics	Solve linear and quadratic equations.	5 a Solve $5 - 3x = 8$. b Solve $(x + 2)^2 = 16$.

1.1 Mappings



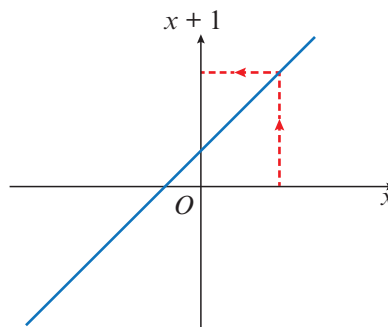
The rule connecting the input and output values can be written algebraically as: $x \mapsto x + 1$.

This is read as 'x is mapped to $x + 1$ '.

The mapping can be represented graphically by plotting values of $x + 1$ against values of x .

The diagram shows that for one input value there is just one output value.

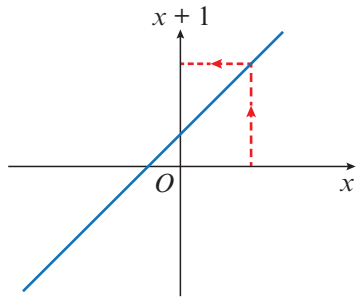
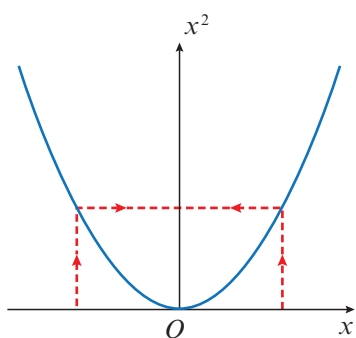
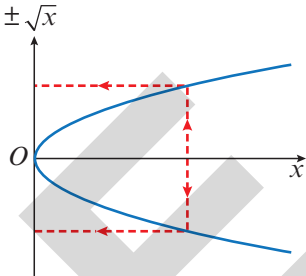
It is called a one-one mapping.



KEY WORDS

mapping diagram
function
one-one function
domain
range
composite function
modulus
absolute value
self-inverse functions

The table below shows one-one, many-one and one-many mappings.

one-one	many-one	one-many
		
For one input value there is just one output value.	For two input values there is one output value.	For one input value there are two output values.

Exercise 1.1

Determine whether each of these mappings is one-one, many-one or one-many.

1 $x \mapsto x + 1 \quad x \in \mathbb{R}$

3 $x \mapsto x^3 \quad x \in \mathbb{R}$

5 $x \mapsto \frac{1}{x} \quad x \in \mathbb{R}, x > 0$

7 $x \mapsto \frac{12}{x} \quad x \in \mathbb{R}, x > 0$

2 $x \mapsto x^2 + 5 \quad x \in \mathbb{R}$

4 $x \mapsto 2^x \quad x \in \mathbb{R}$

6 $x \mapsto x^2 + 1 \quad x \in \mathbb{R}, x \geq 0$

8 $x \mapsto \pm x \quad x \in \mathbb{R}, x \geq 0$

1.2 Definition of a function

A **function** is a rule that maps each x value to just one y value for a defined set of input values. This means that mappings that are either one-one or many-one are called functions.

The mapping $x \mapsto x + 1$, where $x \in \mathbb{R}$, is a **one-one function**.

It can be written as $\begin{cases} f : x \mapsto x + 1 & x \in \mathbb{R} \\ f(x) = x + 1 & x \in \mathbb{R} \end{cases}$

($f : x \mapsto x + 1$ is read as ‘the function f , such that x is mapped to $x + 1$ ’)

$f(x)$ represents the output values for the function f .

So when $f(x) = x + 1$, $f(2) = 2 + 1 = 3$.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

WORKED EXAMPLE 1

$$f(x) = 2x - 1 \quad x \in \mathbb{R}, -1 \leq x \leq 3$$

- Write down the domain of the function f .
- Sketch the graph of the function f .
- Write down the range of the function f .

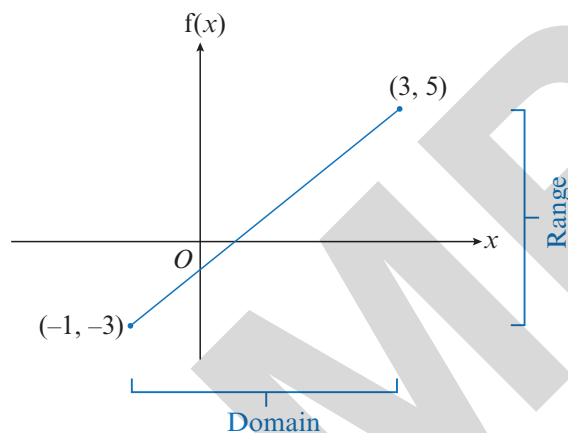
Answers

a The domain of f is $-1 \leq x \leq 3$.

b The graph of $y = 2x - 1$ has gradient 2 and a y -intercept of -1 .

$$\text{When } x = -1, y = 2(-1) - 1 = -3$$

$$\text{When } x = 3, y = 2(3) - 1 = 5$$



c The range of f is $-3 \leq f(x) \leq 5$.

WORKED EXAMPLE 2

The function f is defined by $f(x) = (x - 2)^2 + 3$ for $0 \leq x \leq 6$.

Sketch the graph of the function.

Find the range of f .

Answers

$f(x) = (x - 2)^2 + 3$ is a positive quadratic function, so the graph will be of the form

$$(x - 2)^2 - 3$$

This part of the expression is a square so it will always be ≥ 0 . The smallest value it can be is 0. This occurs when $x = 2$.

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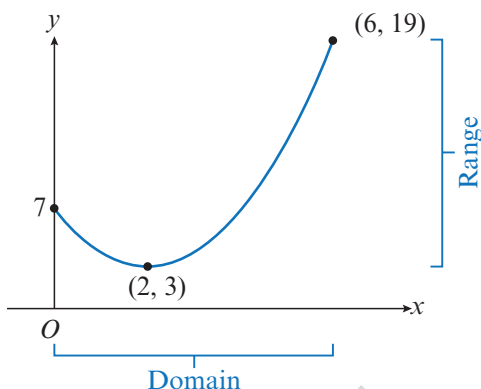
The minimum value of the expression is $0 + 3 = 3$ and this minimum occurs when $x = 2$.

So the function $f(x) = (x - 2)^2 + 3$ will have a minimum point at the point $(2, 3)$.

When $x = 0$, $y = (0 - 2)^2 + 3 = 7$.

When $x = 6$, $y = (6 - 2)^2 + 3 = 19$.

The range of f is $3 \leq f(x) \leq 19$.



Exercise 1.2

- 1 Which of the mappings in **Exercise 1.1** are functions?
- 2 Find the range for each of these functions.

a $f(x) = x - 5$, $-2 \leq x \leq 7$ c $f(x) = 7 - 2x$, $-1 \leq x \leq 4$ e $f(x) = 2^x$, $-3 \leq x \leq 3$	b $f(x) = 3x + 2$, $0 \leq x \leq 5$ d $f(x) = x^2$, $-3 \leq x \leq 3$ f $f(x) = \frac{1}{x}$, $1 \leq x \leq 5$
---	---
- 3 The function g is defined as $g(x) = x^2 + 2$ for $x \geq 0$. Find the range of g .
- 4 The function f is defined by $f(x) = x^2 - 4$ for $x \in \mathbb{R}$. Find the range of f .
- 5 The function f is defined by $f(x) = (x - 1)^2 + 5$ for $x \geq 1$. Find the range of f .
- 6 The function f is defined by $f(x) = (2x + 1)^2 - 5$ for $x \geq -\frac{1}{2}$. Find the range of f .
- 7 The function f is defined by $f : x \mapsto 10 - (x - 3)^2$ for $2 \leq x \leq 7$. Find the range of f .
- 8 The function f is defined by $f(x) = 3 + \sqrt{x - 2}$ for $x \geq 2$. Find the range of f .

1.3 Composite functions

Most functions that you meet are combinations of two or more functions.

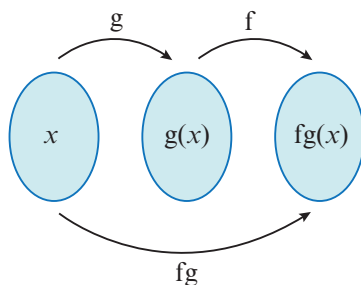
For example, the function $x \mapsto 2x + 5$ is the function ‘multiply by 2 and then add 5’.

It is a combination of the two functions g and f where:

$g : x \mapsto 2x$ (the function 'multiply by 2')

$f : x \mapsto x + 5$ (the function 'add 5')

So, $x \mapsto 2x + 5$ is the function described as 'first do g , then do f '.



When one function is followed by another function, the resulting function is called a **composite function**.

$fg(x)$ means the function g acts on x first, then f acts on the result.

TIP

$f^2(x)$ means $ff(x)$, so you apply the function f twice.

WORKED EXAMPLE 3

The function f is defined by $f(x) = (x - 2)^2 - 3$ for $x > -2$.

The function g is defined by $g(x) = \frac{2x+6}{x-2}$ for $x > 2$.

Find $fg(7)$.

Answers

$fg(7)$

g acts on 7 first and $g(7) = \frac{2(7)+6}{7-2} = 4$

$= f(4)$

f is the function 'take 2, square and then take 3'

$= (4 - 2)^2 - 3$

$= 1$

WORKED EXAMPLE 4

$f(x) = 2x - 1$ for $x \in \mathbb{R}$ $g(x) = x^2 + 5$ for $x \in \mathbb{R}$

Find **a** $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$.

Answers

a $fg(x)$

g acts on x first and $g(x) = x^2 + 5$

$= f(x^2 + 5)$

f is the function 'double and subtract 1'

$= 2(x^2 + 5) - 1$

$= 2x^2 + 9$

CONTINUED

- b** $gf(x)$ f acts on x first and $f(x) = 2x - 1$
 $= g(2x - 1)$ g is the function 'square and add 5'
 $= (2x - 1)^2 + 5$ expand brackets
 $= 4x^2 - 4x + 1 + 5$
 $= 4x^2 - 4x + 6$
- c** $f^2(x)$ $f^2(x)$ means $ff(x)$
 $= ff(x)$ f acts on x first and $f(x) = 2x - 1$
 $= f(2x - 1)$ f is the function 'double and take 1'
 $= 2(2x - 1) - 1$
 $= 4x - 3$

Exercise 1.3

- 1** $f : x \mapsto 2x + 3$ for $x \in \mathbb{R}$
 $g : x \mapsto x^2 - 1$ for $x \in \mathbb{R}$
 Find $fg(2)$.
- 2** $f(x) = x^2 - 1$ for $x \in \mathbb{R}$
 $g(x) = 2x + 3$ for $x \in \mathbb{R}$
 Find $gf(5)$.
- 3** $f(x) = (x + 2)^2 - 1$ for $x \in \mathbb{R}$
 Find $f^2(3)$.
- 4** The function f is defined by $f(x) = 1 + \sqrt{x - 2}$ for $x \geq 2$.
 The function g is defined by $g(x) = \frac{10}{x} - 1$ for $x > 0$.
 Find $gf(18)$.
- 5** The function f is defined by $f(x) = (x - 1)^2 + 3$ for $x > -1$.
 The function g is defined by $g(x) = \frac{2x + 4}{x - 5}$ for $x > 5$.
 Find $fg(7)$.
- 6** $h : x \mapsto x + 2$ for $x > 0$
 $k : x \mapsto \sqrt{x}$ for $x > 0$
 Express each of the following in terms of h and k .
- a** $x \mapsto \sqrt{x} + 2$
b $x \mapsto \sqrt{x + 2}$

- 7** The function f is defined by $f : x \mapsto 3x + 1$ for $x \in \mathbb{R}$.
The function g is defined by $g : x \mapsto \frac{10}{2-x}$ for $x \neq 2$.
Solve the equation $gf(x) = 5$.
- 8** $g(x) = x^2 + 2$ for $x \in \mathbb{R}$
 $h(x) = 3x - 5$ for $x \in \mathbb{R}$
Solve the equation $gh(x) = 51$.
- 9** $f(x) = x^2 - 3$ for $x > 0$
 $g(x) = \frac{3}{x}$ for $x > 0$
Solve the equation $fg(x) = 13$.
- 10** The function f is defined by $f : x \mapsto \frac{3x+5}{x-2}$, $x \neq 2$, for $x \in \mathbb{R}$.
The function g is defined by $g : x \mapsto \frac{x-1}{2}$, for $x \in \mathbb{R}$.
Solve the equation $gf(x) = 12$.
- 11** $f(x) = (x+4)^2 + 3$ for $x > 0$
 $g(x) = \frac{10}{x}$ for $x > 0$
Solve the equation $fg(x) = 39$.
- 12** The function g is defined by $g(x) = x^2 - 1$ for $x \geq 0$.
The function h is defined by $h(x) = 2x - 7$ for $x \geq 0$.
Solve the equation $gh(x) = 0$.
- 13** The function f is defined by $f : x \mapsto x^3$ for $x \in \mathbb{R}$.
The function g is defined by $f : x \mapsto x - 1$ for $x \in \mathbb{R}$.
Express each of the following as a composite function, using only f and/or g :
- a** $x \mapsto (x-1)^3$ **b** $x \mapsto x^3 - 1$ **c** $x \mapsto x - 2$ **d** $x \mapsto x^9$

1.4 Modulus functions

The **modulus** of a number is the magnitude of the number without a sign attached.

The modulus of 4 is written $|4|$.

$$|4| = 4 \text{ and } |-4| = 4$$

It is important to note that the modulus of any number (positive or negative) is always a positive number.

The modulus of a number is also called the **absolute value**.

The modulus of x , written as $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

CLASS DISCUSSION

Ali says that these are all rules for absolute values:

$$|x + y| = |x| + |y|$$

$$|x - y| = |x| - |y|$$

$$|xy| = |x| \times |y|$$

$$\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$$

$$(|x|)^a = x^a$$

Discuss each of these statements with your classmates and decide if they are:

Always true

Sometimes true

Never true

You must justify your decisions.

The statement $|x| = k$, where $k \geq 0$, means that $x = k$ or $x = -k$.

This property is used to solve equations that involve modulus functions.

So, if you are solving equations of the form $|ax + b| = k$, you solve the equations

$$ax + b = k \quad \text{and} \quad ax + b = -k$$

If you are solving harder equations of the form $|ax + b| = cx + d$, you solve the equations

$$ax + b = cx + d \quad \text{and} \quad ax + b = -(cx + d).$$

When solving these more complicated equations, you must always check your answers to make sure that they satisfy the original equation.

WORKED EXAMPLE 5

Solve.

a $|2x + 1| = 5$ **b** $|4x - 3| = x$ **c** $|x^2 - 10| = 6$ **d** $|x - 3| = 2x$

Answers

a $|2x + 1| = 5$

$$2x + 1 = 5 \quad \text{or} \quad 2x + 1 = -5$$

$$2x = 4 \quad 2x = -6$$

$$x = 2 \quad x = -3$$

$$\text{CHECK: } |2 \times 2 + 1| = 5 \checkmark \text{ and } |2 \times -3 + 1| = 5 \checkmark$$

Solution is: $x = -3$ or 2 .

b $|4x - 3| = x$

$$4x - 3 = x \quad \text{or} \quad 4x - 3 = -x$$

$$3x = 3 \quad 5x = 3$$

$$x = 1 \quad x = 0.6$$

$$\text{CHECK: } |4 \times 0.6 - 3| = 0.6 \checkmark \text{ and } |4 \times 1 - 3| = 1 \checkmark$$

Solution is: $x = 0.6$ or 1 .

CONTINUED

c $|x^2 - 10| = 6$

$$x^2 - 10 = 6 \quad \text{or} \quad x^2 - 10 = -6$$

$$x^2 = 16$$

$$x^2 = 4$$

$$x = \pm 4$$

$$x = \pm 2$$

CHECK: $|(-4)^2 - 10| = 6$ ✓, $|(-2)^2 - 10| = 6$ ✓, $|(2)^2 - 10| = 6$ ✓ and $|(4)^2 - 10| = 6$ ✓

Solution is: $x = -4, -2, 2$ or 4 .

d $|x - 3| = 2x$

$$x - 3 = 2x \quad \text{or} \quad x - 3 = -2x$$

$$x = -3$$

$$3x = 3$$

$$x = 1$$

CHECK: $|-3 - 3| = 2 \times -3$ ✗ and $|1 - 3| = 2 \times 1$ ✓

Solution is: $x = 1$.

Exercise 1.4

1 Solve each equation for x .

a $|3x - 2| = 10$

b $|2x + 9| = 5$

c $|6 - 5x| = 2$

d $\left|\frac{x-1}{4}\right| = 6$

e $\left|\frac{2x+7}{3}\right| = 1$

f $\left|\frac{7-2x}{2}\right| = 4$

g $\left|\frac{x}{4} - 5\right| = 1$

h $\left|\frac{x+1}{2} + \frac{2x}{5}\right| = 4$

i $|2x - 5| = x$

2 Solve each equation for x .

a $\left|\frac{2x-5}{x+3}\right| = 8$

b $\left|\frac{3x+2}{x+1}\right| = 2$

c $\left|1 + \frac{x+12}{x+4}\right| = 3$

d $|3x - 5| = x + 2$

e $x + |x - 5| = 8$

f $9 - |1 - x| = 2x$

3 Solve each equation for x .

a $|x^2 - 1| = 3$

b $|x^2 + 1| = 10$

c $|4 - x^2| = 2 - x$

d $|x^2 - 5x| = x$

e $|x^2 - 4| = x + 2$

f $|x^2 - 3| = x + 3$

g $|2x^2 + 1| = 3x$

h $|2x^2 - 3x| = 4 - x$

i $|x^2 - 7x + 6| = 6 - x$

4 Solve each pair of simultaneous equations.

a $y = x + 4$

b $y = x$

c $y = 3x$

$$y = |x^2 - 16|$$

$$y = |3x - 2x^2|$$

$$y = |2x^2 - 5|$$

REFLECTION

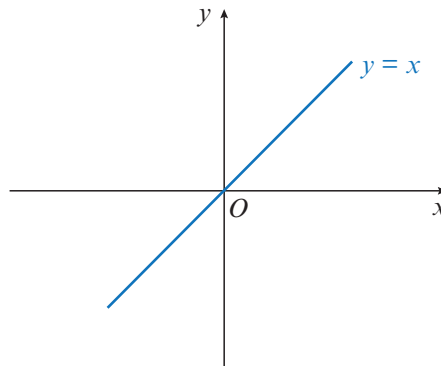
Look back at this section on solving modulus equations.

- 1** What did you find easy?
- 2** What did you find difficult?
- 3** Are there any parts you need to practise more?

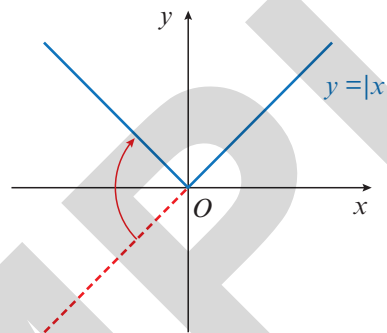
1.5 Graphs of $y = |f(x)|$ where $f(x)$ is linear

Consider drawing the graph of $y = |x|$.

First draw the graph of $y = x$.



Then reflect in the x -axis the part of the line that is below the x -axis.



WORKED EXAMPLE 6

Sketch the graph of $y = \left| \frac{1}{2}x - 1 \right|$,

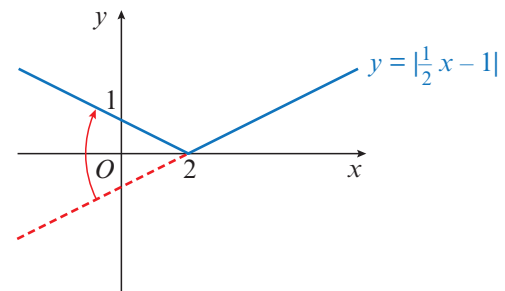
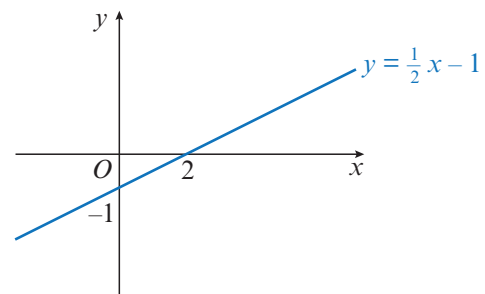
showing the coordinates of the points where the graph intersects the axes.

Answers

First sketch the graph of $y = \frac{1}{2}x - 1$.

The line has gradient $\frac{1}{2}$ and a y -intercept of -1 .

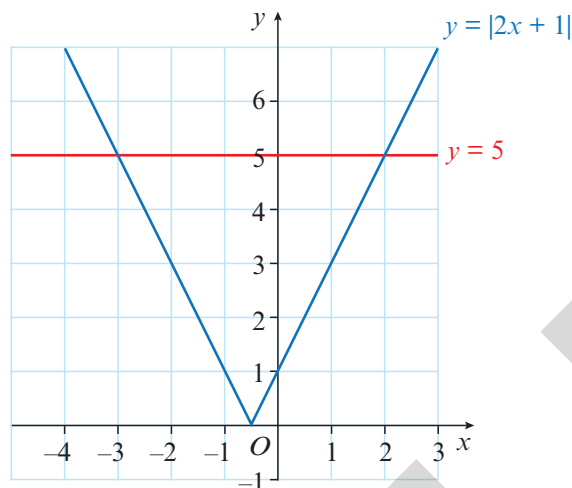
Then reflect in the x -axis the part of the line that is below the x -axis.



In **Worked example 5** you saw that there were two answers, $x = -3$ or $x = 2$, to the equation

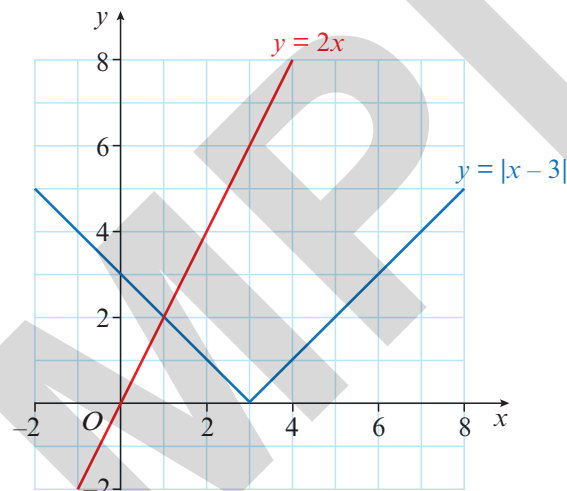
$$|2x + 1| = 5.$$

These can also be found graphically by finding the x -coordinates of the points of intersection of the graphs of $y = |2x + 1|$ and $y = 5$ as shown.



In the same worked example, you also saw that there was only one answer, $x = 1$, to the equation $|x - 3| = 2x$.

This can also be found graphically by finding the x -coordinates of the points of intersection of the graphs of $y = |x - 3|$ and $y = 2x$ as shown.



Exercise 1.5

- 1** Sketch the graphs of each of the following functions, showing the coordinates of the points where the graph intersects the axes.

a $y = |x + 1|$

b $y = |2x - 3|$

c $y = |5 - x|$

d $y = \left| \frac{1}{2}x + 3 \right|$

e $y = |10 - 2x|$

f $y = \left| 6 - \frac{1}{3}x \right|$

- 2 a** Complete the table of values for $y = |x - 2| + 3$.

x	-2	-1	0	1	2	3	4
y		6		4			

- b** Draw the graph of $y = |x - 2| + 3$ for $-2 \leq x \leq 4$.

- 3 Draw the graph of each of the following functions.
- a** $y = |x| + 1$ **b** $y = |x| - 3$ **c** $y = 2 - |x|$
d $y = |x - 3| + 1$ **e** $y = |2x + 6| - 3$
- 4 Given that each of these functions is defined for the domain $-3 \leq x \leq 4$, find the range of
- a** $f : x \mapsto 5 - 2x$ **b** $f : x \mapsto |5 - 2x|$ **c** $h : x \mapsto 5 - |2x|$.
- 5 $f : x \mapsto 3 - 2x$ for $-1 \leq x \leq 4$
 $g : x \mapsto |3 - 2x|$ for $-1 \leq x \leq 4$
 $g : x \mapsto 3 - |2x|$ for $-1 \leq x \leq 4$
Find the range of each function.
- 6 **a** Sketch the graph of $y = |2x + 4|$ for $-6 < x < 2$, showing the coordinates of the points where the graph intersects the axes.
b On the same diagram, sketch the graph of $y = x + 5$.
c Solve the equation $|2x + 4| = x + 5$.
- 7 A function f is defined by $f(x) = |2x - 6| - 3$, for $-1 \leq x \leq 8$.
a Sketch the graph of $y = f(x)$.
b State the range of f .
c Solve the equation $f(x) = 2$.
- 8 **a** Sketch the graph of $y = |3x - 4|$ for $-2 < x < 5$, showing the coordinates of the points where the graph intersects the axes.
b On the same diagram, sketch the graph of $y = 2x$.
c Solve the equation $2x = |3x - 4|$.
- 9 **CHALLENGE QUESTION**
a Sketch the graph of $f(x) = |x + 2| + |x - 2|$.
b Use your graph to solve the equation $|x + 2| + |x - 2| = 6$.

1.6 Inverse functions

The inverse of a function $f(x)$ is the function that undoes what $f(x)$ has done.

The inverse of the function $f(x)$ is written as $f^{-1}(x)$.

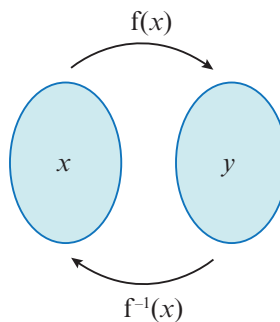
The domain of $f^{-1}(x)$ is the range of $f(x)$.

The range of $f^{-1}(x)$ is the domain of $f(x)$.

It is important to remember that not every function has an inverse.

An inverse function $f^{-1}(x)$ can exist if, and only if, the function $f(x)$ is a one-one mapping.

You should already know how to find the inverse function of some simple one-one mappings.



The steps to find the inverse of the function $f(x) = 5x - 2$ are:

Step 1: Write the function as $y =$ $\rightarrow y = 5x - 2$

Step 2: Interchange the x and y variables. $\rightarrow x = 5y - 2$

Step 3: Rearrange to make y the subject. $\rightarrow y = \frac{x+2}{5}$

Therefore, $f^{-1}(x) = \frac{x+2}{5}$ for $f(x) = 5x - 2$

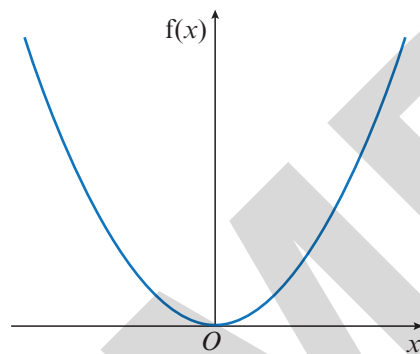
DISCUSSION

Discuss the function $f(x) = x^2$ for $x \in \mathbb{R}$.

Does the function f have an inverse?

Explain your answer.

How could you change the domain of f so that $f(x) = x^2$ does have an inverse?



WORKED EXAMPLE 7

$f(x) = \sqrt{x+1} - 5$ for $x \geq -1$

- Find an expression for $f^{-1}(x)$.
- Solve the equation $f^{-1}(x) = f(35)$.

Answers

a $f(x) = \sqrt{x+1} - 5$ for $x \geq -1$

Step 1: Write the function as $y =$ $\rightarrow y = \sqrt{x+1} - 5$

Step 2: Interchange the x and y variables. $\rightarrow x = \sqrt{y+1} - 5$

Step 3: Rearrange to make y the subject. $\rightarrow x + 5 = \sqrt{y+1}$
 $(x+5)^2 = y+1$
 $y = (x+5)^2 - 1$

$f^{-1}(x) = (x+5)^2 - 1$

CONTINUED

b $f(35) = \sqrt{35+1} - 5 = 1$

$$(x+5)^2 - 1 = 1$$

$$(x+5)^2 = 2$$

$$x+5 = \pm\sqrt{2}$$

$$x = -5 \pm \sqrt{2}$$

$$x = -5 + \sqrt{2} \text{ or } x = -5 - \sqrt{2}$$

The range of f is $f(x) \geq -5$ so the domain of f^{-1} is $x \geq -5$.

Hence the only solution of $f^{-1}(x) = f(35)$ is $x = -5 + \sqrt{2}$.

Exercise 1.6

1 $f(x) = (x+5)^2 - 7$ for $x \geq -5$. Find an expression for $f^{-1}(x)$.

2 $f(x) = \frac{6}{x+2}$ for $x \geq 0$. Find an expression for $f^{-1}(x)$.

3 $f(x) = (2x-3)^2 + 1$ for $x \geq 1.5$. Find an expression for $f^{-1}(x)$.

4 $f(x) = 8 - \sqrt{x-3}$ for $x \geq 3$. Find an expression for $f^{-1}(x)$.

5 $f : x \mapsto 5x - 3$ for $x > 0$ $g : x \mapsto \frac{7}{2-x}$ for $x \neq 2$

Express $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x .

6 $f : x \mapsto (x+2)^2 - 5$ for $x > -2$

a Find an expression for $f^{-1}(x)$.

b Solve the equation $f^{-1}(x) = 3$.

7 $f(x) = (x-4)^2 + 5$ for $x > 4$

a Find an expression for $f^{-1}(x)$.

b Solve the equation $f^{-1}(x) = f(0)$.

8 $g(x) = \frac{2x+3}{x-1}$ for $x > 1$

a Find an expression for $g^{-1}(x)$.

b Solve the equation $g^{-1}(x) = 5$.

9 $f(x) = \frac{x}{2} + 2$ for $x \in \mathbb{R}$ $g(x) = x^2 - 2x$ for $x \in \mathbb{R}$

a Find $f^{-1}(x)$.

b Solve $fg(x) = f^{-1}(x)$.

10 $f(x) = x^2 + 2$ for $x \in \mathbb{R}$ $g(x) = 2x + 3$ for $x \in \mathbb{R}$

Solve the equation $gf(x) = g^{-1}(17)$.

11 $f : x \mapsto \frac{2x+8}{x-2}$ for $x \neq 2$ $g : x \mapsto \frac{x-3}{2}$ for $x > -5$

Solve the equation $f(x) = g^{-1}(x)$.

12 $f(x) = 3x - 24$ for $x \geq 0$. Write down the range of f^{-1} .

13 $f : x \mapsto x + 6$ for $x > 0$ $g : x \mapsto \sqrt{x}$ for $x > 0$

Express $x \mapsto x^2 - 6$ in terms of f and g .

- 14 $f: x \mapsto 3 - 2x$ for $0 \leq x \leq 5$
 $g: x \mapsto |3 - 2x|$ for $0 \leq x \leq 5$
 $h: x \mapsto 3 - |2x|$ for $0 \leq x \leq 5$
 State which of the functions f , g and h has an inverse.
- 15 $f(x) = x^2 + 2$ for $x \geq 0$ $g(x) = 5x - 4$ for $x \geq 0$
a Write down the domain of f^{-1} . **b** Write down the range of g^{-1} .
- 16 The functions f and g are defined, for $x \in \mathbb{R}$, by
 $f: x \mapsto 3x - k$, where k is a positive constant
 $g: x \mapsto \frac{5x-14}{x+1}$ where $x \neq -1$.
a Find expressions for f^{-1} and g^{-1} .
b Find the value of k for which $f^{-1}(5) = 6$.
c Simplify $g^{-1}g(x)$.
- 17 $f: x \mapsto x^3$ for $x \in \mathbb{R}$ $g: x \mapsto x - 8$ for $x \in \mathbb{R}$
 Express each of the following as a composite function, using only f , g , f^{-1} and/or g^{-1} :
a $x \mapsto (x-8)^{\frac{1}{3}}$ **b** $x \mapsto x^3 + 8$ **c** $x \mapsto x^{\frac{1}{3}} - 8$ **d** $x \mapsto (x+8)^{\frac{1}{3}}$

1.7 The graph of a function and its inverse

In **Worked example 1** you considered the function $f(x) = 2x - 1$, $x \in \mathbb{R}$, $-1 \leq x \leq 3$.

The domain of f was $-1 \leq x \leq 3$ and the range of f was $-3 \leq f(x) \leq 5$.

The inverse function is $f^{-1}(x) = \frac{x+1}{2}$.

The domain of f^{-1} is $-3 \leq x \leq 5$ and the range of f^{-1} is $-1 \leq f^{-1}(x) \leq 3$.

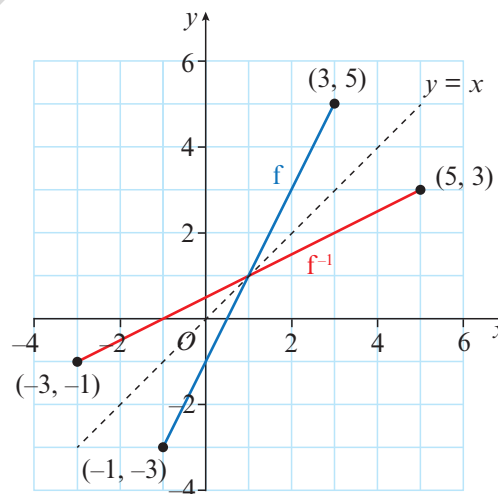
Drawing f and f^{-1} on the same graph gives:

Some functions are called **self-inverse functions** because f and its inverse f^{-1} are the same.

If $f(x) = \frac{1}{x}$ for $x \neq 0$, then $f^{-1}(x) = \frac{1}{x}$ for $x \neq 0$.

So $f(x) = \frac{1}{x}$ for $x \neq 0$ is an example of a self-inverse function.

When a function f is self-inverse, the graph of f will be symmetrical about the line $y = x$.



WORKED EXAMPLE 8

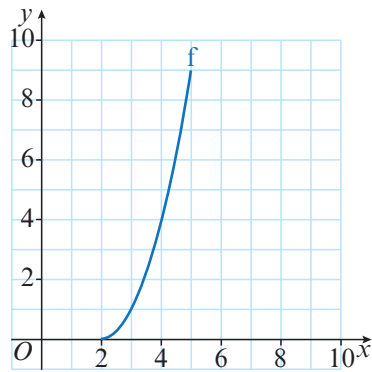
$$f(x) = (x - 2)^2, 2 \leq x \leq 5$$

On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the points where the curves meet the coordinate axes.

Answers

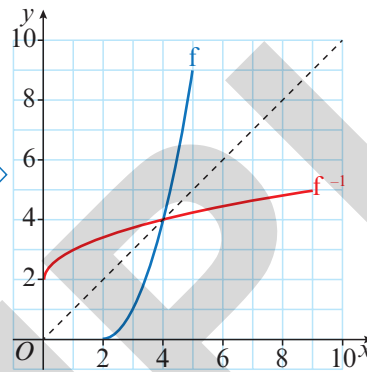
$$y = (x - 2)^2$$

This part of the expression is a square so it will always be ≥ 0 . The smallest value it can be is 0. This occurs when $x = 2$.



When $x = 5$, $y = 9$.

Reflect f in $y = x$



TIP

The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

This is true for all one-one functions and their inverse functions.

This is because:
 $ff^{-1}(x) = x = f^{-1}f(x)$.

DISCUSSION

Sundeep says that the diagram shows the graph of the function $f(x) = x^x$ for $x > 0$, together with its inverse function $y = f^{-1}(x)$.

Is Sundeep correct? Explain your answer.

